Lesson 3.4

Solve linear programming problems

**Vocabulary**

**Optimization** means finding the maximum or minimum value of some quantity.

**Linear programming** is the process of optimizing a linear **objective function** subject to a system of linear inequalities called **constraints**.

The graph of the system of constraints is called the **feasible region**. If an objective function has a maximum or a minimum value, then it must occur at a vertex of the feasible region. Moreover, the objective function will have both a maximum and a minimum value if the feasible region is bounded.

**Example 1**

Find the minimum and maximum values of the objective function \( C = 3x + 2y \) subject to the following constraints.

\[
3x + 4y \leq 20 \\
3x - y \leq 5 \\
x \geq 0 \\
y \geq 0
\]

**Solution**

The feasible region determined by the constraints is shown. The three vertices \((0, 5)\), \((0, 0)\), and \((\frac{8}{3}, 0)\) are intercepts. The fourth vertex \((\frac{8}{3}, 3)\) is found by solving the system of equations \(3x + 4y = 20\) and \(3x - y = 5\). To find the minimum and maximum values of \(C\), evaluate \(C = 3x + 2y\) at each of the four vertices.

At \((0, 5)\): \(C = 3(0) + 2(5) = 10\)

At \((0, 0)\): \(C = 3(0) + 2(0) = 0\)

At \((\frac{8}{3}, 0)\): \(C = 3\left(\frac{8}{3}\right) + 2(0) = 5\)

At \((\frac{8}{3}, 3)\): \(C = 3\left(\frac{8}{3}\right) + 2(3) = 14\)

The minimum value of \(C\) is 0, which occurs when \(x = 0\) and \(y = 0\).

The maximum value of \(C\) is 14, which occurs when \(x = \frac{8}{3}\) and \(y = 3\).
Exercises for Example 1

Find the minimum and maximum values of the objective function subject to the given constraints.

1. Objective function: \( C = 6x - 2y \)
   Constraints: 
   \[
   \begin{align*}
   & x + y \leq 9 \\
   & 4x + y \geq 12 \\
   & x \geq 0 \\
   & y \geq 0
   \end{align*}
   \]

2. Objective function: \( C = x + 3y \)
   Constraints: 
   \[
   \begin{align*}
   & x + y \leq 5 \\
   & x \geq 1 \\
   & y \geq 2
   \end{align*}
   \]

EXAMPLE 2 A Region that Is Unbounded

Find the minimum and maximum values of the objective function \( C = 2x + 5y \) subject to the following constraints.

\[
\begin{align*}
& 3x - 5y \geq 24 \\
& x - y \geq 6 \\
& x \geq 0 \\
& y \leq 0
\end{align*}
\]

Solution

The feasible region determined by the constraints is shown. The two vertices \((8, 0)\) and \((0, -6)\) are intercepts. The third vertex \((3, -3)\) is found by solving the system of equations \(3x - 5y \geq 24\) and \(x - y \geq 6\).

Now evaluate \( C = 2x + 5y \) at each of the three vertices.

At \((0, -6)\): \( C = 2(0) + 5(-6) = -30 \)
At \((8, 0)\): \( C = 2(8) + 5(0) = 16 \)
At \((3, -3)\): \( C = 2(3) + 5(-3) = -9 \)

Since the feasible region has no lower bound, the objective function has no minimum value. The maximum value of \( C \) is 16.

Exercises for Example 2

Find the minimum and maximum values of the objective function subject to the given constraints.

3. Objective function: \( C = -2x + y \)
   Constraints: 
   \[
   \begin{align*}
   & y - x \leq 0 \\
   & x \geq 2 \\
   & y \geq 1
   \end{align*}
   \]

4. Objective function: \( C = 3x - y \)
   Constraints: 
   \[
   \begin{align*}
   & -9x - 4y \geq -16 \\
   & 5x - 3y \geq 35 \\
   & x \geq 0 \\
   & y \leq -5
   \end{align*}
   \]